Axiomatic Quantification of Multidimensional Image Resolution

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***Abstract*—We generalize the axiomatic quantification of one-dimensional (1-D) image resolution to the multidimensional case. The imaging system of interest is characterized by a nonnegative spatially invariant point spread function. The axioms extended from the 1-D counterparts include nonnegativity, continuity, translation invariance, rotation invariance, luminance invariance, homogeneous scaling, and serial combination properties. It is proved that the only resolution measure consistent with the axioms is proportional to the square root of the trace of the covariance matrix of the point spread function.**

***Index Terms*—Axiomatic derivation, image resolution, resolution.**

I. Introduction

In a recent letter [1], Wang and Li presented an axiomatic approach for defining image resolution. For a one-dimensional (1-D), nonnegative, spatially invariant imaging system. They proved that, based on their axioms, the image resolution measure must be the standard deviation of the system point spread function (PSF), up to a constant multiplier. In this letter, we generalize their finding to the multidimensional case. The 1-D axioms are transformed into multidimensional versions, along with a new rotation axiom. We establish that the only resolution measure consistent with these multidimensional axioms is proportional to the square root of the trace of the covariance matrix of the PSF.

II. Axioms

The imaging system under consideration is modeled as

$$i = p * o \tag{1}$$

where o is a function representing an object or a scene, p is a nonnegative spatially invariant PSF, i is an image, and $*$ denotes N-dimensional convolution. For this imaging system (1), let $R[p]$ denote a measure of its image resolution. By convention, the smaller $R[p]$ is, the finer detail the system can resolve.

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In the following analysis, we assume that the nonnegative function p is integrable with finite first- and second-order moments. Let G^{σ} denote a Gaussian probability density function with zero mean vector and covariance matrix σ. Denote an $N \times N$ identity matrix by $\mathbf{I}$.

Similar to what was done in [1], the multidimensional axioms on $R[p]$ are postulated as follows.

Axiom 1 (nonnegativity): $R[p] \geq 0$; $R[G^{\mathbf{I}}] > 0$ and is finite.

Axiom 2 (continuity): $R[p]$ is a continuous function of p in the following sense: if p_n converges to p vaguely in the sense of measure as measure density functions [2, p. 217], then $R[p_n]$ converges to $R[p]$. That is, if for all $f \in C_0(\mathbf{R}^N)$

$$\int_{\mathbf{R}^N} p_n \cdot f \longrightarrow \int_{\mathbf{R}^N} p \cdot f$$

then $R[p_n] \longrightarrow R[p]$.

Axiom 3 (translation invariance): For every $x_0 \in \mathbf{R}^N$, let p_{x_0} be the PSF defined by

$$p_{x_0}(x) = p\,(x - x_0) \quad \forall x \in \mathbf{R}^N$$

then $R[p_{x_0}] = R[p]$.

Axiom 4 (rotation invariance): For every real $N \times N$ orthogonal matrix $\mathbf{T}$, let $p_{\mathbf{T}}$ be the PSF defined by

$$p_{\mathbf{T}}(x) = p(\mathbf{T} \cdot x) \quad \forall x \in \mathbf{R}^N$$

then $R[p_{\mathbf{T}}] = R[p]$.

Axiom 5 (luminance invariance): For all $c > 0$, $R[cp] = R[p]$.

Axiom 6 (scaling): If the argument of p is scaled, the measure of resolution is scaled by the same amount. That is, let $p_1(x) = p(x/\beta)$ for some finite $\beta > 0$, then

$$R\,[p_1] = \beta R[p]. \tag{2}$$

Axiom 7 (combination): There exists a function F such that for any two imaging systems with PSFs p_1 and p_2, respectively, the image resolution measure of the composite system by serial connection of the two systems p_1 and p_2 is

$$R[p] = R\,[p_1 * p_2] = F\,[R\,[p_1]\,, R\,[p_2]]\,. \tag{3}$$

Comments:

1) At least one PSF must have positive resolution measure to rule out the trivial case $R[p] = 0$ for all p.
2) The continuity, translation invariance, luminance invariance, scaling, and combination axioms are essentially the same as 1-D counterparts in Wang and Li [1].

3) Rotation invariance assumes all axes are in the same units and no direction is preferred. Further comments on this axiom are in the discussion section.
4) The combination axiom allows the reduction of a general PSF to a Gaussian distribution with the same covariance matrix as that PSF normalized to unit area. The function F is uniquely determined by the axioms.

III. Derivation

The derivation has two parts. First, the image resolution of a Gaussian distribution is determined. Second, the image resolution for a general PSF is found in terms of a corresponding Gaussian distribution. The following lemma is immediate from the axioms.

Lemma 1: For any integrable function $p(x)$ with finite first moment, under **Axioms 3** and **5**, $R[p] = R[p_1]$, where

$$p_1(x) = \frac{1}{\int p(x)dx} p(x-\mu) \tag{4}$$

and

$$\mu = \frac{1}{\int p(x)dx} \int xp(x)dx. \tag{5}$$

Thus, it is sufficient to consider PSFs $p(x)$ that are probability density functions with mean zero.

Denote the set of real $N \times N$ positive definite symmetric matrices by $\mathcal{P}$ and the set of real $N \times N$ symmetric matrices by $\mathcal{S}$. The closure of $\mathcal{P}$ is $\bar{\mathcal{P}}$, the set of nonnegative definite symmetric matrices.

A. Gaussian Case

Let G^σ be the N-dimensional Gaussian PSF with mean vector $\mu = (0, \ldots, 0)^{\mathbf{tr}} = \mathbf{0} \in \mathbf{R}^N$ and covariance matrix $\sigma = (\sigma_{j,k}) \in \mathcal{P}$

$$G^\sigma(x) = \frac{1}{(2\pi)^{N/2} (\det(\sigma))^{1/2}} \mathrm{e}^{-(1/2)x^{\mathbf{tr}} \cdot \sigma^{-1} \cdot x} \quad \text{for } x \in \mathbf{R}^N \tag{6}$$

where $\det(\sigma)$ is the determinant of σ. For the standard normalized Gaussian $G^{\sigma=\mathbf{I}}$, we assume that its resolution is α; that is, $R[G^{\mathbf{I}}] = \alpha$.

From the convolution of two Gaussians

$$G^{\sigma_1} * G^{\sigma_2} = G^{\sigma_1+\sigma_2}. \tag{7}$$

Lemma 1 implies that the resolution of any Gaussian depends only on σ. Denote the function of a Gaussian that determines resolution based on the covariance matrix by f

$$R[G^\sigma] = f(\sigma). \tag{8}$$

Based on the combination **Axiom 7** and (7), we have the following properties of F and f: $f(\sigma_1 + \sigma_2) = F(f(\sigma_1), f(\sigma_2))$ and $\alpha = f(\mathbf{I})$.

Let $\langle \cdot, \cdot \rangle$ denote the inner product of matrices, $\langle \mathbf{C}, \mathbf{Q} \rangle = \mathrm{Trace}(\mathbf{C}^{\mathbf{tr}}\mathbf{Q})$.

Theorem 2: Under **Axioms 1–5, 6,** and **7**, the following properties for F and f hold:

$$f(\lambda\sigma) = \sqrt{\lambda} f(\sigma), \quad \text{for } \lambda > 0 \text{ and } \sigma \in \mathcal{P} \tag{9}$$

$$F(x,y) = \sqrt{x^2+y^2}, \quad \text{for } x, y > 0 \tag{10}$$

$$f(\sigma) = \sqrt{\langle \mathbf{C}, \sigma \rangle}, \quad \text{for } \sigma \in \mathcal{P} \tag{11}$$

where $\mathbf{C} \in \mathcal{S}$ is a symmetric matrix.

Proof: **Axiom 6** states that scaling x by $1/\beta$ results in a change of resolution by a factor of β. For Gaussians, multiplying x by $1/\beta$ is equivalent to multiplying σ by β^2. This yields (9). In particular, $f((x^2/\alpha^2)\mathbf{I}) = x$ for $x > 0$. From the properties for F and f

$$F(x,y) = F\left(f\left(\frac{x^2}{\alpha^2}\mathbf{I}\right), f\left(\frac{y^2}{\alpha^2}\mathbf{I}\right)\right) \tag{12}$$

$$= f\left(\frac{x^2+y^2}{\alpha^2}\mathbf{I}\right) = \sqrt{x^2+y^2} \tag{13}$$

proving (10). Next, consider f^2. It satisfies the equalities

$$\begin{aligned} f^2(\sigma_1+\sigma_2) &= F^2(f(\sigma_1), f(\sigma_2)) \\ &= f^2(\sigma_2) + f^2(\sigma_1) \end{aligned} \tag{14}$$

and $f^2(\lambda\sigma) = \lambda f^2(\sigma)$. Thus, f^2 is homogeneous and additive on $\mathcal{P}$.

Now we extend f^2 from $\mathcal{P}$ to $\mathcal{S}$. Any $\mathbf{Q} \in \mathcal{S}$ can be written as the difference between two positive definite matrices, such as $\mathbf{Q} = \sigma_1 - \sigma_2$, where $\sigma_1, \sigma_2 \in \mathcal{P}$. Define a new function g by $g(\mathbf{Q}) = f^2(\sigma_1) - f^2(\sigma_2)$. To show that g is well-defined we must show that if it is also true that $\mathbf{Q} = \sigma_3 - \sigma_4$, then $f^2(\sigma_1) - f^2(\sigma_2) = f^2(\sigma_3) - f^2(\sigma_4)$. This follows since the equality $\sigma_1 + \sigma_4 = \sigma_2 + \sigma_3$ and (14) imply that $f^2(\sigma_1) + f^2(\sigma_4) = f^2(\sigma_2) + f^2(\sigma_3)$. To prove that g coincides with f^2 on $\mathcal{P}$, first note that $\lim_{\lambda \to 0+} f(\lambda I) = 0$ by (9). Secondly, any $\mathbf{Q} \in \mathcal{P}$ can be written as $\mathbf{Q} = \sigma_1 - \sigma_2$ with $\sigma_1 = \lambda I + \mathbf{Q} \in \mathcal{P}$ and $\sigma_2 = \lambda I \in \mathcal{P}$ for any $\lambda > 0$. Then $g(\mathbf{Q}) = f^2(\lambda I + \mathbf{Q}) - f^2(\lambda I)$. Letting $\lambda \to 0+$, we have $g(\mathbf{Q}) = f^2(\mathbf{Q})$, by **Axiom 2** and the limit property of f^2 near $0I$.

Next, we show that g is a linear functional on S. For any $Q_1, Q_2 \in \mathcal{S}$, let $Q_1 = \sigma_1 - \sigma_2$ and $Q_2 = \sigma_3 - \sigma_4$ with $\sigma_i \in \mathcal{P}$, $i = 1, 2, 3, 4$. Note now $Q_1 + Q_2 = (\sigma_1 + \sigma_3) - (\sigma_2 + \sigma_4)$ is an admissible decomposition for the definition of g. $g(Q_1 + Q_2) = g(Q_1) + g(Q_2)$ is equivalent to $f^2(\sigma_1 + \sigma_3) - f^2(\sigma_2 + \sigma_4) = f^2(\sigma_1) - f^2(\sigma_2) + f^2(\sigma_3) - f^2(\sigma_4)$ by the definition of g, which follows from $f^2(\sigma_1 + \sigma_3) + f^2(\sigma_2) + f^2(\sigma_4) = f^2(\sigma_1) + f^2(\sigma_3) + f^2(\sigma_2 + \sigma_4)$ by (14). This immediately implies that g is additive on $\mathcal{S}$. To show that g is homogeneous, consider $g(\lambda\mathbf{Q})$. For $\lambda > 0$, we have $g(\lambda\mathbf{Q}) = \lambda g(\mathbf{Q})$ easily from the definition of g and the corresponding property for f (9). For $\lambda < 0$, $g(\lambda\mathbf{Q}) = g(-\lambda(-\mathbf{Q})) = -\lambda g(-\mathbf{Q})$. Finally, if $\mathbf{Q} = \sigma_1 - \sigma_2$, $\sigma_1, \sigma_2 \in \bar{\mathcal{P}}$, then $g(-\mathbf{Q}) = f^2(\sigma_2) - f^2(\sigma_1) = -g(\mathbf{Q})$. Thus, g is a linear functional on $\mathcal{S}$, with a unique representation

$$g(\mathbf{Q}) = \langle \mathbf{C}, \mathbf{Q} \rangle, \quad \text{where } \mathbf{C} \in \mathcal{S}. \tag{15}$$

Equation (11) follows immediately. ∎

Theorem 3: Under **Axioms 1–7**

$$f^2(\sigma) = \frac{\mathrm{Trace}(\sigma)\alpha^2}{N}, \quad \text{for } \sigma \in \mathcal{P}. \tag{16}$$

Proof: Theorem 2 implies $f^2(\sigma) = \langle \mathbf{C}, \sigma\rangle$. **Axiom 4** gives

$$f^2\left(\mathbf{T}\sigma\mathbf{T}^{\mathbf{tr}}\right) = f^2(\sigma) \quad (17)$$

for all orthogonal matrices $\mathbf{T}$. Then

$$\langle \mathbf{C}, \sigma\rangle = \text{Trace}\left(\mathbf{C}^{\mathbf{tr}}\sigma\right) \quad (18)$$
$$= \text{Trace}\left(\mathbf{C}^{\mathbf{tr}}\mathbf{T}\sigma\mathbf{T}^{\mathbf{tr}}\right) \quad (19)$$
$$= \text{Trace}\left(\mathbf{T}^{\mathbf{tr}}\mathbf{C}^{\mathbf{tr}}\mathbf{T}\sigma\right) \quad (20)$$

for all orthogonal matrices $\mathbf{T}$ and all $\sigma \in \bar{\mathcal{P}}$ and by the property that Trace(AB) = Trace(BA). Since $\mathbf{C}$ is symmetric, there exists an orthogonal matrix $\mathbf{T_1}$ such that

$$\mathbf{T}_1^{\mathbf{tr}}\mathbf{C}^{\mathbf{tr}}\mathbf{T}_1 = \mathbf{D} \quad (21)$$

is diagonal, with diagonal entries $\{d_k, k = 1, \ldots, N\}$. Select any pair of diagonal entries d_k and d_l and let $\mathbf{T_2}$ be the permutation matrix that interchanges the kth and lth rows of the identity matrix and leaves the rest untouched. Applying (20) for $T = T_1$ and $T = T_1T_2$

$$\text{Trace}\left(\mathbf{T}_1^{\mathbf{tr}}\mathbf{C}^{\mathbf{tr}}\mathbf{T}_1\sigma\right) = \text{Trace}\left(\mathbf{T}_2^{\mathbf{tr}}\mathbf{T}_1^{\mathbf{tr}}\mathbf{C}^{\mathbf{tr}}\mathbf{T}_1\mathbf{T}_2\sigma\right) \quad (22)$$

for all σ, which implies that

$$d_k\sigma_{k,k} + d_l\sigma_{l,l} = d_l\sigma_{k,k} + d_k\sigma_{l,l} \quad (23)$$

for all nonnegative pairs $(\sigma_{k,k}, \sigma_{l,l})$. This is only possible if $d_k = d_l$. Thus, the diagonal entries of $\mathbf{D}$ must all be equal. Since $f(\mathbf{I}) = \alpha$, we get $d_k = \alpha^2/N$ for all k. ■

Since each $\sigma \in \bar{\mathcal{P}}$ is the limit of a sequence of $\sigma_n \in \mathcal{P}$ and the corresponding PSF G^{σ_n} converges to G^σ vaguely as in the continuity **Axiom 2**, the following corollary follows immediately from Theorem 3.

Corollary : $R\left[G^\sigma\right] = \text{Trace}(\sigma)\alpha^2/N$ for $\sigma \in \bar{\mathcal{P}}$.

B. General Case

Theorem 4: If $R[p]$ is an image resolution measure satisfying **Axioms 1–7**, then for any nonzero nonnegative PSF p

$$R[p] = \frac{\sqrt{\text{Trace}(\sigma)} \cdot \alpha}{\sqrt{N}} \quad (24)$$

where $\sigma = (\sigma_{j,k})$ is the covariance matrix, defined by

$$\sigma_{j,k} = \frac{\int_{\mathbf{R}^N}(x_j - \mu_j)(x_k - \mu_k) \cdot p(x)dx}{\int_{\mathbf{R}^N} p(x)dx} \quad (25)$$

where $\mu = (\mu_1, \ldots, \mu_N)$ is the mean value vector, defined in (5).

Proof: By Lemma 1, we may assume that $\int_{\mathbf{R}^N} p(x)dx = 1$ and $\mu = 0$ without loss of generality. For $n = 1, \ldots$, let $q_n(x) = (\sqrt{n})^N \cdot p(\sqrt{n} \cdot x)$ and $g_n = q_n{}^* q_n{}^* \cdots q_n$, where there are n terms. Then by **Axioms 5** and **6**, $R[q_n] = R[p(t)]/\sqrt{n}$. By **Axiom 7** and Theorem 2, $R[g_n] = R[p(t)]$.

Let $\hat{f}$ denote the Fourier transform of f, defined as

$$\hat{f}(\omega) = \int_{\mathbf{R}^N} f(x) \cdot \mathbf{e}^{-iw\cdot x}dx.$$

For $\hat{p}$, we have, for ω small enough, by a Taylor series expansion around $\omega = 0$ (following Doob [3, p. 140])

$$\hat{p}(\omega) = 1 - \frac{1}{2}\omega^{\mathbf{tr}} \cdot \sigma \cdot \omega + o\left(\|\omega\|^2\right).$$

Since $\hat{q}_n(\omega) = \hat{p}(\omega/\sqrt{n})$, we have

$$\hat{q}_n(\omega) = 1 - \frac{1}{2n}\omega^{\mathbf{tr}} \cdot \sigma \cdot \omega + o\left(\frac{\|\omega\|^2}{n}\right)$$

for any $\omega \in \mathbf{R}^N$, when n is large enough. Then

$$\hat{g}_n(\omega) = \hat{q}_n^n = \left(1 - \frac{1}{2n}\omega^{\mathbf{tr}} \cdot \sigma \cdot \omega + o\left(\frac{\|\omega\|^2}{n}\right)\right)^n.$$

Hence

$$\lim_{n\to\infty} \hat{g}_n(\omega) = \mathbf{e}^{-(1/2)\omega^{\mathbf{tr}}\cdot\sigma\cdot\omega} = \widehat{G^\sigma}(\omega) \quad (26)$$

for any $\omega \in \mathbf{R}^N$. By [2, Prop. 8.69], g_n converges to G^σ vaguely in the sense of measure as measure density functions. Since $R[g_n] = R[p]$ for all n, by the continuity **Axiom 2**, $R[p] = R[G^\sigma]$. The conclusion follows immediately. ■

IV. Discussion

The combination axiom reduces the general case to the Gaussian case. Subject to translation invariance, the resolution of a Gaussian is determined by the covariance matrix. Rotation invariance essentially states that the resolution measure must be basis-free. Selection of different orthonormal axes in N dimensions to represent an image corresponds to a rotation of the covariance matrix, from σ to $\mathbf{T}^{\mathbf{tr}}\sigma\mathbf{T}$, where $\mathbf{T}$ is an orthogonal matrix. Thus, the resolution measure can only depend on the eigenvalues of σ. From (7), the resolution measure must depend on a function of a linear measure of the covariance matrix and thus, on the trace.

If a measure of resolution in a lower dimensional subspace is needed, then the rotational invariance axiom should be abandoned. With the other axioms in place, the measure of resolution still depends on a linear function of the covariance matrix, which is determined by a matrix $\mathbf{C}$ [see (11)]. An approach consistent with the axioms is to project the covariance matrix onto the lower dimensional subspaceand have a resolution measure proportional to the square root of the trace of the resulting lower dimensional covariance matrix. For 1-D resolution in the direction of $\mathbf{u}$, where $\mathbf{u}^{\mathbf{tr}}\mathbf{u} = 1$, let $\mathbf{C} = \mathbf{u}\mathbf{u}^{\mathbf{tr}}$. The resulting measure of resolution is proportional to $\sqrt{\mathbf{u}^{\mathbf{tr}}\sigma\mathbf{u}}$, the standard deviation in that direction.

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