1 Simple Binary Tests

1.1 Exponential Random Variables in Queuing

In queuing systems, packets or messages are processed by blocks in the system. These processing blocks are often called queues. A common model for a queue is that the time it takes to process a message is an exponential random variable. There may be an additional model for the times at which messages enter the queue, a common model of which is a Poisson process.

Recall that if $X$ is exponentially distributed with mean $\mu$, then

$$p_X(x) = \frac{1}{\mu} e^{-\frac{x}{\mu}}, x \geq 0. \quad (1)$$

Suppose that one queue is being monitored. A message enters at time $t = 0$ and exits at time $t = T$. Under Hypothesis $H_1$, $T$ is an exponentially distributed random variable with mean $\mu_1$; under $H_0$, $T$ is exponentially distributed with mean $\mu_0$. Assume that $\mu_1 > \mu_0$.

a. Prove that the likelihood ratio test is equivalent to comparing $T$ to a threshold $\gamma$.

b. For an optimum Bayes test, find $\gamma$ as a function of the costs and the a priori probabilities.

c. Now assume that a Neyman-Pearson test is used. Find $\gamma$ as a function of the bound on the false alarm probability $P_F$, where $P_F = P(\text{false alarm})$.

d. Plot the ROC for this problem for $\mu_0 = 1$ and $\mu_1 = 5$.

e. Now consider $N$ independent and identically distributed measurements of $T$ denoted $T_1, T_2, \ldots, T_N$. Show that the likelihood ratio test may be reduced to comparing

$$l(T) = \frac{1}{N} \sum_{i=1}^{N} T_i \quad (2)$$

to a threshold. Find the probability density function for $l(T)$ under each hypothesis.

1.2 Gaussian Variance

In many problems in radar, the reflectivity is a complex Gaussian random variable. Sequential measurements of a given target that fluctuates rapidly may yield independent realizations of these random variables. It may then be of interest to decide between two models for the variance.

Assume that $N$ independent measurements are made, with resulting i.i.d. random variables $R_i$, $i = 1, 2, \ldots, N$. The models are

$$H_1 : R_i \sim \mathcal{N}(0, \sigma_1^2), i = 1, 2, \ldots, N, \quad (3)$$

$$H_0 : R_i \sim \mathcal{N}(0, \sigma_0^2), i = 1, 2, \ldots, N, \quad (4)$$
where $\sigma_1 > \sigma_0$.

a. Find the likelihood ratio test.
b. Show that the likelihood ratio test may be simplified to comparing the sufficient statistic

$$l(R) = \frac{1}{N} \sum_{i=1}^{N} R_i^2$$

(5)
to a threshold.
c. Find an expression for the probability of false alarm, $P_F$, and the probability of miss, $P_M$.
d. Plot the ROC for $\sigma_0^2 = 1$, $\sigma_1^2 = 2$, and $N = 2$.

1.3 Binary Observations

Suppose that there are only two possible outcomes of an experiment; call the outcomes heads and tails. The problem here is to decide whether the process used to generate the outcomes is fair. The hypotheses are

$$H_1 : \quad P(R_i = \text{heads}) = p, \quad i = 1, 2, \ldots, N$$

(6)

$$H_0 : \quad P(R_i = \text{heads}) = 0.5, \quad i = 1, 2, \ldots, N.$$  

(7)

Under each hypothesis, the random variables $R_i$ are i.i.d.

a. Determine the optimal likelihood ratio test. Show that the number of heads is a sufficient statistic.
b. Note that the sufficient statistic does not depend on $p$, but the threshold does. For a finite number $N$, if only nonrandomized tests are considered, then the ROC has $N+1$ points on it. For $N = 10$ and $p = 0.7$, plot the ROC for this problem. You may want to do this using a computer because you will need the cumulative distribution function for a binomial.
c. Now consider a randomized test. In a randomized test, for each value of the sufficient statistic, the decision is random. Hypothesis $H_1$ is chosen with probability $\phi(l)$ and $H_0$ is chosen with probability $1 - \phi(l)$. Consider the Neyman-Pearson criterion with probability of false alarm $P_F = \alpha$. Show that the optimal randomized strategy is a probabilistic mixture of two ordinary likelihood ratio tests: the first likelihood ratio test achieves the next greater probability of false alarm than $\alpha$, while the second achieves the next lower probability of false alarm than $\alpha$. Find $\phi$ as a function of $\alpha$. Finally, note that the resulting ROC results from the original ROC after connecting the achievable points with straight lines.

2 Likelihood Ratio as a Random Variable

The problem follows Problem 2.2.13 from H. L. Van Trees, Vol. 1, closely.

The likelihood ratio $\Lambda(R)$ is a random variable

$$\Lambda(R) = \frac{p(R|H_1)}{p(R|H_0)}$$

(8)

Prove the following properties of the random variable $\Lambda$.

a. $E(\Lambda^n|H_1) = E(\Lambda^{n+1}|H_0)$
b. $E(\Lambda|H_0) = 1$.
c. $E(\Lambda|H_1) - E(\Lambda|H_0) = \text{var}(\Lambda|H_0)$.

3 Matlab Problem

Download the matlab files posted under “Rough Matlab Files.” Put them into a directory for this class and run matlab from that directory. Note whatever changes you make to these files since updated versions of them may be posted on the web site as we go.
Generate a simulated data set for the sum of exponentials in Gaussian noise. Change the SNR to get several plots. In particular, solve the following problem:

\[ H_0 : \quad R_k = \exp(-k/4) + W_k, \quad k = 1, 2, \ldots, K, \quad (9) \]
\[ H_1 : \quad R_k = \exp(-k/4) + \exp(-k/(4T)) + W_k, \quad k = 1, 2, \ldots, K, \quad (10) \]

where the \( W_k \) are i.i.d. \( \mathcal{N}(0, \sigma^2) \).

\[ \text{a. Plot the performance for fixed } \sigma^2 \text{ and } K \text{ as } T \text{ varies (} T > 1 \).} \]
\[ \text{b. Plot the performance for fixed } T \text{ and } K \text{ as } \sigma^2 \text{ varies.} \]
\[ \text{c. Plot the performance for fixed } \sigma^2 \text{ and } T \text{ as } K \text{ varies.} \]

Evaluate and summarize your results in a concise manner.